HW 6 SOLUTIONS

SECTION 2.4

(2) $4y'' - 4y' + y = 16e^{t/2} \iff y'' - y' + \frac{1}{4}y = \underbrace{4e^{t/2}}_{:= g}$

HOM. SOLN FIRST: $4r^2 - 4r + 1 = 0$

$r = 1/2$ w/ MULTIPLICITY 2

(i.e. REPEATED ROOT)

$\Rightarrow y_1 = e^{t/2}, \quad y_2 = te^{t/2}$

$\Rightarrow y_H = c_1 y_1 + c_2 y_2 = c_1 e^{t/2} + c_2 t e^{t/2}.$

NOW WE NEED y_p: $W(y_1, y_2) = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & e^{t/2} + \frac{t}{2}e^{t/2} \end{vmatrix}$

$= e^t$

$y_p = u_1 e^{t/2} + u_2 t e^{t/2}$

$u_1 = -\int \frac{y_2 \cdot g}{W} dt = -\int \frac{4e^t t}{e^t} dt = -2t^2.$

$u_2 = +\int \frac{y_1 g}{W} dt = \int \frac{4e^t}{e^t} dt = 4t.$

(2) $\Rightarrow y_p = -2t^2 e^{t/2} + 4t \cdot t e^{t/2}$

$= +2t^2 e^{t/2}$

$\therefore \boxed{y_{Gen} = y_H + y_p = c_1 e^{t/2} + c_2 t e^{t/2} + 2t^2 e^{t/2}}$

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(3) $y'' + 9y = 9\sec^2(3t)$, $0 < t < \pi/6$.

$y_{Gen} = y_H + y_p$

$y_H'' + 9y_H = 0 \Rightarrow r^2 + 9 = 0$

$\Rightarrow r = \pm 3i$

$y_H = c_1\cos(3t) + c_2\sin(3t)$

$$W(y_1, y_2) = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\left(\sin^2(3t) + \cos^2(3t)\right)$$

$= 3$

$$y_p = -\cos(3t) \cdot \int \frac{\sin(3t) \cdot 9\sec^2(3t)}{3} dt + \sin(3t) \int \frac{\cos(3t) \cdot 9\sec^2(3t)}{3} dt$$

$$= -3\cos(3t) \cdot \int \frac{\sin(3t)}{\cos^2(3t)} dt + 3\sin(3t) \int \frac{\cos(3t)}{(1 - \sin^2(3t))} dt$$

(3)

$$\int \frac{\sin(3t)}{\cos^2(3t)}\,dt = -\frac{1}{3}\cdot\int \frac{1}{u^2}\,du = +\frac{1}{3}\cdot\frac{1}{u}$$

$$= \frac{1}{3}\sec(3t).$$

$$\int \frac{\cos(3t)}{\left(1-\sin^2(3t)\right)}\,dt = \frac{1}{3}\cdot\int \frac{1}{1-u^2}\,du = (\star)$$

$$\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u} \Rightarrow A(1+u) + B(1-u) = 1$$

$$\Rightarrow \begin{array}{l} A+B = 1 \\ A-B = 0 \end{array} \Rightarrow A = B = 1/2.$$

$$\Rightarrow (\star) = \frac{1}{3}\int\left[\frac{1}{2}\cdot\frac{1}{1-u} + \frac{1}{2}\,\frac{1}{1+u}\right]du = \left[\frac{-1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u)\right]\frac{1}{3}$$

$$= \left[\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)\right](1/3)$$

$$= \left[\frac{1}{2}\ln\left[\frac{1+\sin(3t)}{1-\sin(3t)}\right]\right]1/3.$$

$$\Rightarrow y_p = \cancel{\quad} -3\cos(3t)\cdot\frac{1}{3}\sec(3t)$$

$$+\ 3\sin(3t)\cdot\frac{1}{3}\cdot\frac{1}{2}\ln[\ \cdot\ ] = -1 + \frac{1}{2}\ln\left(\frac{1+\sin(3t)}{1-\sin(3t)}\right)\sin(3t)$$

$$\therefore\ \boxed{y_{Gen} = c_1\cos(3t) + c_2\sin(3t) - 1 + 1/2\,\sin(3t)\cdot\ln\left(\frac{1+\sin(3t)}{1-\sin(3t)}\right)}$$

(5) $t^2 y'' - t(t+2)y' + (t+2)y = 2t^3 \quad (*)$

$y_1 = t, \quad y_2 = te^t$

PLUG IN y_1, y_2 INTO $(*)$ TO CONFIRM THEY SATISFY THE O.D.E..

$$y_p = -y_1 \int \frac{y_2 g}{W}\,dt + y_2 \cdot \int \frac{y_1 g}{W}\,dt$$

WHERE $g = \frac{2t^3}{t^2} = 2t$. AND $W(y_1,y_2) = \begin{vmatrix} t & te^t \\ 1 & e^t + te^t \end{vmatrix}$

$= t^2 e^t$

$$\Rightarrow y_p = -t \cdot \int \frac{te^t \cdot 2t}{t^2 e^t}\,dt + te^t \int \frac{t \cdot 2t}{t^2 e^t}\,dt$$

$$= -t \cdot \int 2\,dt + te^t \int 2e^{-t}\,dt$$

$$= -2t^2 + te^t(-2e^{-t})$$

$$= -2t^2 \underbrace{-2t}_{}$$

CAN DISCARD THIS AS IT'S A CONSTANT MULTIPLE OF y_1

$\therefore \boxed{y_p = -2t^2}$

(6) $(1-x)\,y'' + y = 0, \quad x_0 = 0.$

LET $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} a_n n \cdot x^{n-1}, \; y'' = \sum_{n=2} a_n \cdot n(n-1) \cdot x^{n-2}$

PLUG INTO O.D.E. :

$$\underbrace{\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}}_{\text{LET } K=n-2} - \underbrace{\sum_{n=2}^{\infty} a_n n(n-1) x^{n-1}}_{\text{LET } K=n-1} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\text{LET } K=n} = 0.$$

$\rightarrow$ EQUATION BECOMES:

$$\sum_{K=0}^{\infty} a_{K+2}(K+2)(K+1)x^K - \sum_{K=1}^{\infty} a_{K+1}(K+1)(K)x^K + \sum_{K=0}^{\infty} a_K x^K = 0$$

$\Longrightarrow$

$$\left[a_2(2)(1) + a_0\right] + \sum_{K=1}^{\infty}\left[a_{K+2}(K+2)(K+1) - a_{K+1}(K+1)(K) + a_K\right]x^K = 0$$

NOW REQUIRE EACH TERM IN THE SERIES EQUAL 0.

$\Rightarrow$

$K=0: \; 2a_2 + a_0 = 0$

$K=1: \; (3)(2)a_3 - 2a_2 + a_1 = 0$

$K=2: \; (4)(3)a_4 - (3)(2)a_3 + a_2 = 0$

SECTION 2.8

2

$$\sum_{n=0}^{\infty} \underbrace{2^n x^n}_{\equiv a_n} .$$

USE RATIO TEST TO DETERMINE R.O.C. :

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} \frac{x^{n+1}}{x^n} = 2x$$

$$\Rightarrow \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |2x|$$

WHICH WE NEED STRICTLY LESS THAN 1.

$\Rightarrow$ SERIES CONVERGES WHEN $|2x| < 1$

$$\Updownarrow$$

$$-1/2 < x < 1/2$$

$$\therefore \boxed{\text{R.O.C.} = 1/2}$$

FOR GENERAL $k \geq 1$:

$$(k+2)(k+1)a_{k+2} - (k+1)(k)a_{k+1} + a_k = 0$$

$$\Rightarrow \boxed{a_{k+2} = \frac{(k+1)(k)a_{k+1} - a_k}{(k+2)(k+1)}}$$

RECURRENCE RELATION

(b)

$$a_2 = -\tfrac{1}{2} a_0$$

$$a_3 = \frac{1}{6}(2a_2 - a_1) = \frac{1}{6}(-a_0 - a_1)$$

$$= -\frac{1}{6}a_0 - \frac{1}{6}a_1$$

$$a_4 = \frac{1}{12}(6a_3 - a_2) = \frac{1}{2}\left(-\frac{1}{6}a_0 - \frac{1}{6}a_1\right) - \frac{1}{12}\left(-\frac{1}{2}a_0\right)$$

$$= -\frac{1}{12}a_0 - \frac{1}{12}a_1 + \frac{1}{24}a_0$$

$$= -\frac{1}{24}a_0 - \frac{1}{12}a_1$$

RECALL:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x + \left(-\frac{1}{2}a_0\right)x^2 + \left(-\frac{1}{6}a_0 - \frac{1}{6}a_1\right)x^3$$

$$+ \left(-\frac{1}{24}a_0 - \frac{1}{12}a_1\right)x^4 + \cdots$$

$$\Rightarrow \boxed{y_{Gen} = a_0 \overbrace{\left(1 - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 + \ldots\right)}^{:= y_1} + a_1 \underbrace{\left(x - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 + \ldots\right)}_{:= y_2}}$$

(c)

AT $x_0 = 0$, $y_1 = 1$, $y_1' = 0$

$y_2 = 0$, $y_2' = 1$

$$\Rightarrow W(y_1, y_2)(x_0) = y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0)$$

$$= (1)(1) - (0)(0)$$

$$= 1 \neq 0$$

SO WRONSKIAN IS <u>NOT</u> <u>IDENTICALLY</u> ZERO $\Rightarrow$ y_1, y_2 FORM FUNDAMENTAL SOL$^{\underline{n}}$ SET.

(d) 

9. $y'' + \underbrace{4}_{\equiv p(x)} y' + \underbrace{6x}_{\equiv q(x)} y = 0$, $x_0 = 0$, $x_0 = 4$

$p(x) = 4$ IS ANALYTIC EVERYWHERE

$q(x) = 6x$ IS ALSO ANALYTIC EVERYWHERE.

$\Rightarrow$ SERIES SOL^{n} WILL HAVE INFINITE RADIUS OF CONVERGENCE

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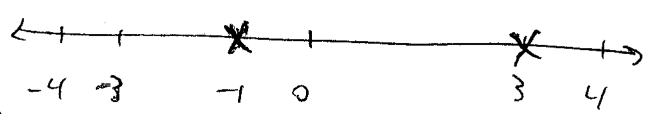
10. $(x^2 - 2x - 3)y'' + xy' + 4y = 0$

$x_0 = 4$, $x_0 = -4$, $x_0 = 0$

$$\rightarrow \quad y'' + \underbrace{\left(\frac{x}{x^2 - 2x - 3}\right)}_{\equiv p(x)} y' + \underbrace{\left(\frac{4}{x^2 - 2x - 3}\right)}_{\equiv q(x)} y = 0$$

$p(x)$ DISCONTINUOUS AT $x = 3, -1$

$q(x)$ DISCONTINUOUS AT $x = 3, -1$



SOL^{n} AT $x_0 = 4$:
$R = |4 - 3| = 1$
R.O.C. AT LEAST 1

SOL^{n} AT $x_0 = -4$:
$R = |-4 - (-1)| = 3$
R.O.C. AT LEAST 3

SOL^{n} AT $x_0 = 0$
$R = |0 - (-1)| = 1$
R.O.C. AT LEAST 1